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APPLICATIONS OF THE INTEGRATION BY PARTS FORMULA I

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ABSTRACT

In the present work we have established some applications of the integration by parts formula which enables to lay the foundations of the study of regularity properties of the distributions of the solution process of stochastic delay equation.

Keywods:- *Stochastic Differential Equations, Malliavin Calculus, Euler Scheme for delay SDE's, Integration by Parts, Densities of Distributions.*

I. INTRODUCTION, NOTATION AND DEFINITIONS

1.1 Introduction

In Chapter 1 of the Ph.D. thesis of Ahmed [15] we have proved the existence and uniqueness of a solution for certain types of delay (functional) stochastic differential equations (delay SDE's) with discontinuous initial data,see also [1], [9] and the web cite www.sfde.math.siu.edu. See the delay SDE (1.1) in the present work. In [18] we have established integration by parts formula involving Mallivan derivatives of solutions to such type of delay (functional) SDE's. The integration by parts formula which we establish can be used to extend the formulas in [2] and [3] and to include delay SDE's as well as ordinary SDE's. In this work we also establish some other useful applications to delay SDE's. Generally speaking we can say that our work extends the first three chapters of the work by Norris to include delay SDE's as well as ordinary SDE's; see Theorems 2.3, 3.1 and 3.2 in [10] . In a sequal paper we will show that that the distribution of the solution process has smooth densities. Moreover we will establish integration by part formula involving Malliavin derivatives of higher order.

1.2 Notations and Definitions

The following notations and definitions will be used throughout this work: $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space; T is a positive real number; $\{\mathcal{F}_t\}_{t\in[0,T]}$ is an increasing family of sub-σ algebras of $\mathcal{F}$, each of which contains all null subsets of Ω; $\mathbb{N}$ is the set of natural numbers; $W = (W^1, \dots, W^r): [0,T] \times \Omega \to \mathbb{R}^r$ is a r-dimensional normalized Brownian motion. If X is a topological space, then $\mathcal{B}(X)$ denotes its Borel field. The symbol λ refers to the Lebesgue measure on $\mathbb{R}^d$, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$, $d \in \mathbb{N}$.

Let G be a Banach space and let $\mathcal{A}$ be a sub-σ algebra of $\mathcal{F}$ containing all subsets of measure zero in $\mathcal{F}$, then $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P}; G)$ denotes the space of all functions $f: \Omega \to G$ which are $\mathcal{A}$-$\mathcal{B}(G)$ measurable and are such that $\int_\Omega \|f\|_G^2 \, d\mathbb{P} < \infty$.

The symbol $L^2(\Omega, \mathcal{A}, \mathbb{P}; G)$ denotes the Banach space (with norm determined by $\|f\|_{L^2}^2 = \int_\Omega \|f(\omega)\|_G^2 \, d\mathbb{P}$) of all equivalence classes of functions $f: \Omega \to G$ which are $\mathcal{A}$-$\mathcal{B}(G)$ measurable and which are such that

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$\int_{\Omega}\|f\|_G^2\, d\mathbb{P} < \infty$. The symbol $L(\mathbb{R}^m, \mathbb{R}^n)$ $(m, n \in \mathbb{N})$ denotes the space of all linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. The symbol J refers to the interval $[-1,0)$, and $\mathcal{H}(J)$ or $\mathcal{B}(J)$ refers to the Borel field on J.

If $X: [-1,T] \times \Omega \to \mathbb{R}^d$ is a process, then for each $t \in [0,T]$ and $\omega \in \Omega$ we define the map: $X_t: \Omega \to \mathcal{L}^2(J, \mathbb{R}^d)$ by $X_t(\omega)(s) = X(t+s, \omega)$ for all $s \in J$ and almost all ω. For each $0 \le t \le T$ we write $\|(X(t), X_t)\|^2 = \|X(t)\|^2 + \|X_t\|^2$. Let the function V belong to $\mathcal{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$, θ belong to $\mathcal{L}^2(J \times \Omega, \mathcal{H}(J) \otimes \mathcal{F}_0, \lambda \otimes \mathbb{P}; \mathbb{R}^d)$, and for $\ell = 1,2, \dots, r$ let f ,g^ℓ be functions from $[0,T] \times \Omega \times \mathbb{R}^d \times \mathcal{L}^2(J, \mathbb{R}^d)$ to $\mathbb{R}^d$. Then a process $X: [-1,T] \times \Omega \to \mathbb{R}^d$ is called a solution of the delay SDE with integral form

$$X(t) = \begin{cases} V + \int_0^t f(u, X(u), X_u)du + \sum_{\ell=1}^r \int_0^t g^\ell(u, X(u), X_u)\, dW^\ell(u), & 0 \le t \le T, \\ \theta(t), & t \in J, \end{cases} \tag{1.1}$$

If

(i) X is $\mathcal{B}([0,T]) \otimes \mathcal{F}$-$\mathcal{B}(\mathbb{R}^d)$ measurable;

(ii) For each$t \in [0,T]$, the process $X(t,\cdot)$ is $\mathcal{F}_t$-$\mathcal{B}(\mathbb{R}^d)$ measurable, and for each $t \in J$, the process $X(t,\cdot)$ is $\mathcal{F}_0$-$\mathcal{B}(\mathbb{R}^d)$ measurable;

(iii)$X \in \mathcal{L}^2([-1,T] \times \Omega, \mathcal{H} \times \mathcal{F}, \lambda \times \mathbb{P}; \mathbb{R}^d)$,

(IV) X satisfies the delay SDE ([1.1.1]).

The following conditions are sufficient for the existence of a unique solution to (1.1) (see [1] and [15]).

(i) $V \in \mathcal{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$.

(ii) $\theta \in \mathcal{L}^2(J \times \Omega, \mathcal{H} \otimes \mathcal{F}_0, \lambda \otimes \mathbb{P}, \mathbb{R}^d)$.

(iii) $f, g^\ell: [0,T] \times \Omega \times \mathbb{R}^d \times \mathcal{L}^2(J, \mathbb{R}^d) \to \mathbb{R}^d$ are such that

(a) f andg^ℓ are $\mathcal{B}([0,T]) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(L^2(J, \mathbb{R}^d))$-$\mathcal{B}(\mathbb{R}^d)$ measurable.

(b) For each $t \in [0,T]$, the stochastic variables $f(t,\cdot,\cdot,\cdot)$ and $g^\ell(t,\cdot,\cdot,\cdot)$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathcal{L}^2(J, \mathbb{R}^d))$-$\mathcal{B}(\mathbb{R}^d)$ measurable.

(c)There exists a constant K and a function $\zeta \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ such that $|f(t,\omega,s,h)| + \sum_{\ell=1}^r |g^\ell(t,\omega,s,h)| \le K(|s| + \|h\| + |\zeta(\omega)|)$ (1.2) (1.2)

for almost all ω and for all $t \in [0,T]$; $s \in \mathbb{R}^d$ and h belongs to $\mathcal{L}^2(J, \mathbb{R}^d)$.

(d) There exists a constant K' such that, for almost all ω,

$$\begin{aligned} |f(t,\omega,s,h_1) - f(t,\omega,u,h_2)| + \sum_{\ell=1}^r |g^\ell(t,\omega,s,h_1) - g^\ell(t,\omega,u,h_2)| \\ \le K'(|s-u| + \|h_1 - h_2\|) \end{aligned} \tag{1.3}$$

for all $t \in [0,T]$; for all $s, u \in \mathbb{R}^d$, and for all $h_1, h_2 \in \mathcal{L}^2(J, \mathbb{R}^d)$.

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II. INTEGRATION BY PARTS FORMULA

In the beginning of this section we recall the following eight basic numbered equations and definitions, See(16) and(17) . For $(X(0), X_0) = (x, \xi) \in \mathbb{R}^d \times L^2(J, \mathbb{R}^d)$, let $v \mapsto D^v X^{x,\xi}(t)$, be the Malliavin derivative of the solution process $X^{x,\xi}(t)$. We write $D^v X_t^{x,\xi}(\vartheta) = D^v X^{x,\xi}(t+\vartheta)$ $(t \in [0,T], \vartheta \in J = [-1,0))$ for its time delay.

In the following definition we give a precise definition of the Malliavin derivative of a real-valued functional F of Brownian motion.

I.Definition: Let $F((W(s))_{0\le s\le T})$ be a functional of r-dimensional Brownian motion, and let $v(t) = (v^1(t), \ldots, v^r(t))^* = \begin{pmatrix} v^1(t) \\ \vdots \\ v^r(t) \end{pmatrix}$ be a deterministic vector-valued function in $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$. Then $D^v F((W(s))_{0\le s\le T})$ is given by the limit:

$$D^v F((W(s))_{0\le s\le T}) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(F\left(\left(W(s) + \varepsilon \int_0^s v(\sigma)\,d\sigma\right)_{0\le s\le s}\right) - F((W(s))_{0\le s\le t})\right). \quad (2.1)$$

The mapping $v \mapsto D^v F((W(s))_{0\le s\le T})$ is a linear map (functional) from the space $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$to $\mathbb{R}$. Here $\mathbb{R}^r \otimes \mathbb{R}^d$ denotes the space of all $r \times d$-matrices (r rows, d columns).

Notice that, for $v(t) = (v^1(t), \ldots, v^r(t))^\tau = \begin{pmatrix} v^1(t) \\ \vdots \\ v^r(t) \end{pmatrix}$ be a deterministic matrix-valued function in $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$, $U^v(t)$ can be considered as a $d \times d$-matrix where each entry is an $\mathbb{R}$-valued adapted stochastic process; U_t^v can be considered as a $d \times d$-matrix where each entry is an $L^2(J, \mathbb{R})$-valued adapted stochastic process. If $M = (m_{jk})_{1\le j\le d,\, 1\le k\le r}$ is a real $d \times r$ matrix, then $M^\tau = (m_{kj})_{1\le k\le r,\, 1\le j\le d}$ denotes its transposed: it is $r \times d$ matrix with entries m_{kj}.

The process $D^v X_t^{x,\xi}(\cdot)$ satisfies the following delay stochastic differential equation:

$$\begin{aligned} dD^v X_t(\vartheta) &= dD^v X(t+\vartheta) \\ &= \left(\frac{\partial f}{\partial x}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) D^v X(t+\vartheta)\right. \\ &\quad \left.+ \int_J \frac{\partial f}{\partial \xi}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})(\varphi) D^v X_{t+\vartheta}(\varphi)\,d\varphi\right) dt \\ &\quad + \sum_{\ell=1}^r \frac{\partial g^\ell}{\partial x}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) D^v X(t+\vartheta)\,dW^\ell(t+\vartheta) \\ &\quad + \sum_{\ell=1}^r \int_J \frac{\partial g^\ell}{\partial \xi}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})(\varphi) D^v X_{t+\vartheta}(\varphi)\,d\varphi\,dW^\ell(t+\vartheta) \\ &\quad + \sum_{\ell=1}^r g^\ell(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) v^\ell(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})\,dt, \end{aligned} \quad (2.2)$$

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where ϑ belongs to J. If $t+\vartheta$ belongs to J we replace $t+\vartheta$ with 0 in(2.2). If $\vartheta=0$ we obtain the delay stochastic differential equation for the process $D^v X(t)$:

$$\begin{aligned} &dD^v X(t) \\ &= \left(\frac{\partial f}{\partial x}(t, X(t), X_t) D^v X(t) + \int_J \frac{\partial f}{\partial \xi}(t, X(t), X_t)(\vartheta) D^v X_t(\vartheta) d\vartheta\right) dt \\ &+ \sum_{\ell=1}^r \left(\frac{\partial g^\ell}{\partial x}(t, X(t), X_t) D^v X(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t, X(t), X_t)(\vartheta) D^v X_t(\vartheta) d\vartheta\right) dW^\ell(t) \\ &+ \sum_{\ell=1}^r g^\ell(t, X(t), X_t) v^\ell(t, X(t), X_t) dt. \end{aligned} \quad (2.3)$$

We also write $U_{11}^{x,\xi}(t)=\frac{\partial}{\partial x}X^{x,\xi}(t)$, and $U_{12}^{x,\xi}(t)=\frac{\partial}{\partial \xi}X^{x,\xi}(t)$. In addition, we write $U_{21}^{x,\xi}(t)=\frac{\partial}{\partial x}X_t^{x,\xi}=U_{11,t}^{x,\xi}$ (the delay of $U_{11}^{x,\xi}(t)$), and $U_{22}^{x,\xi}(t)=\frac{\partial}{\partial \xi}X_t^{x,\xi}=U_{12,t}^{x,\xi}$, the delay of the process $U_{12}^{x,\xi}(t)$. The matrix $U_{11}^{x,\xi}(t)$ can be identified with an operator from $\mathbb{R}^d$ to itself, the matrix $U_{12}^{x,\xi}(t)$ can be considered as an linear mapping from $L^2(J,\mathbb{R}^d)$ to $\mathbb{R}^d$, the matrix $U_{21}^{x,\xi}(t)$ as a mapping from $\mathbb{R}^d$ to $L^2(J,\mathbb{R}^d)$, and, finally, $U_{22}^{x,\xi}(t)$ as a mapping from $L^2(J,\mathbb{R}^d)$ to itself. Notice that $U_{11}^{x,\xi}(t)$ can be considered as $d\times d$-matrix where each entry is an $\mathbb{R}$-valued adapted stochastic process; $U_{12}^{x,\xi}(t)$ can be considered as $d\times d$-matrix where each entry is an $L^2(J,\mathbb{R})$-valued adapted stochastic process. To be precise, write the solution process as a d-vector $X^{x,\xi}(t)=\left(X_1^{x,\xi}(t),\dots,X_d^{x,\xi}(t)\right)$, and consider the mapping ($1\le j,k\le d$)

$$\xi_k \to X_j^{x,(\xi_1,\dots,\xi_{k-1},\xi_k,\xi_{k+1},\dots,\xi_d)}(t), \quad (2.4)$$

which is a mapping from $L^2(J,\mathbb{R})$ to $\mathbb{R}$, and where each variable ξ_ℓ, , $l\neq k$, is a fixed function in $L^2(J,\mathbb{R})$. The derivative of the function in (2.4) can be considered as a continuous linear functional on $L^2(J,\mathbb{R})$. Therefore it can be represented as an inner-product with a function in $L^2(J,\mathbb{R})$, which is denoted by $\frac{\partial x_j^{x,\xi}(t)}{\partial \xi_k}$. Consequently, we write

$$\begin{aligned} \frac{\partial x_j^{x,\xi}(t)}{\partial \xi_k}(\eta) &= \lim_{h\to 0}\frac{x_j^{x,(\xi_1,\dots,\xi_{k-1},\xi_k+h\eta,\xi_{k+1},\dots,\xi_d)}(t)-x_j^{x,(\xi_1,\dots,\xi_{k-1},\xi_k,\xi_{k+1},\dots,\xi_d)}(t)}{h} \\ &= \int_J \eta(\varphi)\frac{\partial x_j^{x,\xi}(t)}{\partial \xi_k}(\varphi)d\varphi, \quad \eta\in L^2(J,\mathbb{R}). \end{aligned} \quad (2.5)$$

After giving a brief introduction to our work, we are now ready to continue the work that we have started in (16).

Here, and in the sequel, we write $f(t)$ and $g^\ell(t)$ instead of $f\left(t, X^{x,\xi}(t), X_t^{x,\xi}\right)$ and $g^\ell\left(t, X^{x,\xi}(t), X_t^{x,\xi}\right)$

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respectively. For a concise formulation of the stochastic differential equation for the matrix-valued process $(U(t): t \geq)$ and its inverse we introduce the following *stochastic differentials*:

$$h_x(t) = \frac{\partial f}{\partial x}(t)d\,t + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial x}(t)\,d\,W^\ell(t); \qquad (2.6)$$

$$h_\xi(t) = \frac{\partial f}{\partial \xi}(t)d\,t + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial \xi}(t)\,d\,W^\ell(t) \qquad (2.7)$$

$$h_\xi(t,\vartheta) = \frac{\partial f}{\partial \xi}(t,\vartheta)d\,t + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)\,d\,W^\ell(t) (2.8)$$

Application of the Integration by Parts Formula:

Relevant SDE's are ($v(t)$ is a $r \times d$ matrix-valued adapted process: d columns, r rows)

$$dD^vX(t) = h_x(t)D^vX(t) + \int_J h_\xi\,(t,\vartheta)D^vX_t(\vartheta)\,d\vartheta + \sum_{\ell=1}^{r} g^\ell\,(t)v^\ell(t)^\tau\,dt; \; (2.9)$$

$$dV^v(t) = -V^v(t)\,h_x(t) - V^v(t)\int_J h_\xi\,(t,\vartheta)D^vX_t(\vartheta)\,d\vartheta\big(D^vX(t)\big)^{-1}$$
$$+V^v(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell(t)}{\partial x} + \int_J \frac{\partial g^\ell(t,\vartheta)}{\partial \xi}D^vX_t(\vartheta)\,d\vartheta\big(D^vX(t)\big)^{-1}\right)^2 dt; \; (2.10)$$

$$d\,U^v(t) = h_x(t)\,U^v(t) + \int_J h_\xi\,(t,\vartheta)D^vX_t(\vartheta)\,d\vartheta\big(D^vX(t)\big)^{-1}U^v(t). \qquad (2.11)$$

We have to arrange things in such a way that each $g^\ell(t)$ is an $d \times 1$-matrix, i.e. a column of height d. The matrix $v^\ell(t)$ is a $1 \times d$-matrix, i.e. a column vector of length$d$, which is predictable. For $v^\ell(t) = 0$, $1 \leq \ell \leq r$, and $D^0X(0) = I$, the process $D^vX(t) = D^0X(t)$ can be identified with the process $U_{11}(t)$, i.e. the space flow of the solution process $X(t)$. Below we write $U_{11}(t)$ instead of $U(t)$.

A candidate choice for the vector $v^\ell(t)$ is the process$v^\ell(t) = V(t)g^\ell(t)$, where

$$dU(t) = h_x(t)\,U(t) + \int_J h_\xi\,(t,\vartheta)U_t(\vartheta)d\vartheta; \qquad (2.12)$$

$$dV(t) = -V(t)h_x(t) - V(t)\int_J h_\xi\,(t,\vartheta)U_t(\vartheta)d\vartheta U(t)^{-1}$$
$$+V(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta U(t)^{-1}\right)^2 dt. \,(2.13)$$

If, in addition, $U(0)V(0) = I$, then it follows that $U(t)V(t) = I$; see the proof of the following Theorem 2.

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2 Theorem. Suppose that the stochastic vectors $v^{\ell}(t)$, $1 \leq \ell \leq r$ are chosen in such a way that the Malliavin derivatives $D^{v}X(t)$, $0 < t < \infty$ are invertible. Also suppose that $U^{v}(0)V^{v}(0) = I$. Then

$$V^{v}(t)D^{v}X(t) = V^{v}(0)D^{v}X(0) + \int_0^t V^{v}(s)\sum_{\ell=1}^{r} g^{\ell}(s)v^{\ell}(s)^{\tau}\,ds \quad \text{and} \quad (2.14)$$

$$U^{v}(t)V^{v}(t) = U^{v}(0)V^{v}(0) = I. \quad (2.15)$$

In particular, if $D^{v}X(0) = 0$, then

$$D^{v}X(t) = U^{v}(t)\int_0^t V^{v}(s)\sum_{\ell=1}^{r} g^{\ell}(s)v^{\ell}(s)^{\tau}\,ds. \quad (2.16)$$

If, moreover, $v^{\ell}(t) = g^{\ell}(t)^{\tau}V(t)^{\tau}$, then

$$D^{v}X(t) = U^{v}(t)\int_0^t V^{v}(s)\sum_{\ell=1}^{r} g^{\ell}(s)g^{\ell}(s)^{\tau}V(s)^{\tau}\,ds. \quad (2.17)$$

Proof. The first of these equalities follows from a straight forward application of Itô's lemma. More precisely we have:

$$d\left(V^{v}(\cdot)D^{v}X(\cdot)\right)(t)$$

$$= dV^{v}(t)D^{v}X(t) + V^{v}(t)dD^{v}X(t) + d < V^{v}(\cdot), D^{v}X(\cdot) > (t)$$

$$= -V^{v}(t)h_x(t)D^{v}X(t) - V^{v}(t)\int_J h_{\xi}(t,\vartheta)D^{v}X_t(\vartheta)d\vartheta$$

$$+V^{v}(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^{\ell}}{\partial x}(t)\int_J \frac{\partial g^{\ell}}{\partial \xi}(t,\vartheta)D^{v}X_t(\vartheta)d\vartheta\left(D^{v}X(t)\right)^{-1}\right)^{2} D^{v}X(t)dt$$

$$+V^{v}(t)h_x(t)D^{v}X(t) + V^{v}(t)\int_J h_{\xi}(t,\vartheta)D^{v}X_t(\vartheta)d\vartheta$$

$$+V^{v}(t)\sum_{\ell=1}^{r} g^{\ell}(t)v^{\ell}(t)^{\tau}dt$$

$$-V^{v}(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^{\ell}}{\partial x}(t) + \int_J \frac{\partial g^{\ell}}{\partial \xi}(t,\vartheta)D^{v}X_t(\vartheta)d\vartheta(D^{v}X(t))^{-1}\right)$$
$$\left(\frac{\partial g^{\ell}}{\partial x}(t)D^{v}X(t) + \int_J \frac{\partial g^{\ell}}{\partial \xi}(t,\vartheta)D^{v}X_t(\vartheta)d\vartheta\right)dt$$

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$$= V^v(t)\sum_{\ell=1}^{r} g^{\ell}(t)v^{\ell}(t)^{\tau}dt. \ (2.18)$$

This proves (2.14). The second equality is based on Itô's formula in conjunction with Gronwall's lemma:

$$d(U^\wedge v(\cdot)V^\wedge v(\cdot))(t)$$

$$= dU^v(t)V^v(t) + U^v(t)dV^v(t) + d < U^v(\cdot), V^v(\cdot) > (t)$$

$$= h_x(t)U^v(t)V^v(t) + \int_J h_\xi(t,\vartheta)D^vX_t(\vartheta)\, d\vartheta\left(D^vX(t)\right)^{-1}U^v(t)V^v(t)$$

$$-U^v(t)V^v(t)h_x(t) - U^v(t)V^v(t)\int_J h_\xi(t,\vartheta)D^vX_t(\vartheta)\, d\vartheta\left(D^vX(t)\right)^{-1}$$

$$+U^v(t)V^v(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^{\ell}(t)}{\partial x} + \int_J \frac{\partial g^{\ell}(t,\vartheta)}{\partial \xi}D^vX_t(\vartheta)\, d\vartheta(D^vX(t))^{-1}\right)^2 dt$$

$$-\sum_{\ell=1}^{r}\left(\frac{\partial g^{\ell}}{\partial x}(t) + \int_J \frac{\partial g^{\ell}}{\partial \xi}(t)D^vX_t(\vartheta)d\vartheta(D^vX(t))^{-1}\right)U^v(t)V^v(t)$$

$$\left(\frac{\partial g^{\ell}}{\partial x}(t) + \int_J \frac{\partial g^{\ell}}{\partial \xi}(t)D^vX_t(\vartheta)d\vartheta(D^vX(t))^{-1}\right)dt. \qquad (2.19)$$

Put $B^v(t) = U^v(t)V^v(t) - U^v(0)V^v(0)$. From (2.19) we infer:

$$\begin{aligned} dB^v(t) \quad &= \left(h_x(t) + \int_J h_\xi(t,\vartheta)D^vX_t(\vartheta)d\vartheta\left(D^vX(t)\right)^{-1}\right)B^v(t) \\ &\quad -B^v(t)\left(h_x(t) + \int_J h_\xi(t,\vartheta)D^vX_t(\vartheta)d\vartheta\left(D^vX(t)\right)^{-1}\right) \\ &\quad +B^v(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^{\ell}(t)}{\partial x} + \int_J \frac{\partial g^{\ell}(t,\vartheta)}{\partial \xi}D^vX_t(\vartheta)\, d\vartheta\left(D^vX(t)\right)^{-1}\right)^2 dt \\ &\quad -\sum_{\ell=1}^{r}\left(\frac{\partial g^{\ell}}{\partial x}(t) + \int_J \frac{\partial g^{\ell}}{\partial \xi}(t)D^vX_t(\vartheta)d\vartheta\left(D^vX(t)\right)^{-1}\right)B^v(t) \\ &\quad \left(\frac{\partial g^{\ell}}{\partial x}(t) + \int_J \frac{\partial g^{\ell}}{\partial \xi}(t)D^vX_t(\vartheta)d\vartheta(D^vX(t))^{-1}\right) dt. \ (2.20) \end{aligned}$$

Since $B^v(0) = 0$, from (2.20) the second claim in Theorem 2 follows.

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From (2.14) and (2.15) we easily infer:

$$D^v X(t) = U^v(t)V^v(0)D^v X(0) + U^v(t)\int_0^t V^v(s)\sum_{\ell=1}^{r} g^\ell(s)v^\ell(s)^\tau\, ds. \quad (2.21)$$

Consider the following system of delay stochastic differential equations:

$$\begin{cases} dF_j(t) = h_x(t)F_j(t) + \int_J h_\xi(t,\vartheta)F_{j,t}(\vartheta)d\vartheta + \sum_{\ell=1}^{r} g^\ell(t)\left(g^\ell(t)\right)^\tau V_j(t)^\tau dt; \\ F_j(0) = 0 \quad \text{and} \quad V_j(0) = V_{j+1}(0) = I; \\ V_{j+1}(t)F_j(t) = \int_0^t V_{j+1}(s)\sum_{\ell=1}^{r} g^\ell(s)\left(g^\ell(s)\right)^\tau V_j(s)^\tau ds. \end{cases} \quad (2.22)$$

If we set $v_j^\ell(t) = (g^\ell(t))^\tau V_j(t)^\tau$, then $F_j(t) = D^{v_j}X(t)$, the Malliavin derivative taken in the direction of $v_j(t)$, of the solution process. In principle we could in an inductive manner find a sequence $\left(V_j(t), F_j(t)\right)$, $j = 0$, 1, 2, ...r, where the pairs $\left(V_j(t), F_j(t)\right)$ and $\left(V_{j+1}(t), F_{j+1}(t)\right)$ satisfy the first equation in (2.22), and where the pair $\left(V_{j+1}(t), F_j(t)\right)$ satisfies the second equation of (2.22). The limit would yield a solution to the following system of equations:

$$\begin{cases} dF_0(t) = h_x(t)F_0(t) + \int_J h_\xi(t,\vartheta)F_{0,t}(\vartheta)d\vartheta + \sum_{\ell=1}^{r} g^\ell(t)\left(g^\ell(t)\right)^\tau V_0(t)^\tau dt; \\ F_0(0) = 0 \quad \text{and} \quad V_0(0) = I; \\ V_0(t)F_0(t) = \int_0^t V_0(s)\sum_{\ell=1}^{r} g^\ell(s)\left(g^\ell(s)\right)^\tau V_0(s)^\tau ds. \end{cases} \quad (2.23)$$

In Theorem 3 below we show the existence of the pair $(V_0(t), F_0(t))$ as a limit of the sequence $\left(V_{j+1}(s), F_j(s)\right)$. Put $v_0^\ell(t) = (g^\ell(t))^\tau V_0(t)^\tau$, where M^τ indicates the transposed of the matrix M. Then $F_0(t) = D^{v_0}X(t)$, and $F_0(t)^{-1}$ belongs to L^p for all $1 \le p < \infty$ if and only if this is true for the inverse of the Malliavin covariance matrix $\left(\int_0^t V_0(s)\sum_{\ell=1}^{r} g^\ell(s)(g^\ell(s))^*(V_0(s))^* ds\right)^{-1}$. This kind of difficulty does not pose itself if there is no delay. If there is no delay, then in (2.23) for $V_0(t)$ we may take the inverse of the “standard flow":
$U_0(t) = U(t) = \frac{\partial}{\partial x}X^{x,\xi}(t)$.

3 Theorem. The system of delay stochastic differential equation (2.23)possesses a unique solution.

Proof. We will employ an iterative method to construct a convergent sequence of space flows $\left(V_j : j \in \mathbb{N}\right)$. Once $V_j(t)$ is constructed, then, as remarked above, $F_j(t) = D^{v_j}X(t)$, the Malliavin derivative of the solution process

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$X(t)$ in the direction of $v_j(t) := \sum_{\ell=1}^{r}(g^{\ell}(t))^{\tau}\left(V_j(t)\right)^{\tau}$. In order to solve the second equation in (2.23) for the process $V_{j+1}(t)$ we define the process $C_j(s)$ by the following identity (cf(2.22))

$$\frac{1}{s}F_j(s) = \sum_{\ell=1}^{r} g^{\ell}(s)\left(g^{\ell}(s)\right)^{\tau}\left(V_j(s)\right)^{\tau}\left(I + sC_j(s)\right)^{-1}. \qquad (2.24)$$

Put $A_{j+1}(t) = V_{j+1}(t)F_j(t)$. The second equation in (2.22) can then be rewritten as:

$$A_{j+1}(t) = \int_0^t A_{j+1}(s)\left(\frac{1}{s}I + C_j(s)\right)ds, \qquad (2.25)$$

or, what amounts to the same $(0 < \varepsilon < t)$,

$$A_{j+1}(t) = A_{j+1}(\varepsilon) + \int_{\varepsilon}^{t} A_{j+1}(s)\left(\frac{1}{s}I + C_j(s)\right)ds. \qquad (2.26)$$

An iteration of the equality in (2.26) yields:

$$\begin{aligned}
&A_{j+1}(t) - A_{j+1}(\varepsilon)\\
&= A_{j+1}(\varepsilon)\left(\sum_{k=1}^{n}\int_{\varepsilon<s_1<\cdots<s_k<t}\int\left(\frac{1}{s_1}I + C_j(s_1)\right)\cdots\left(\frac{1}{s_k}I + C_j(s_k)\right)ds_k \ldots ds_1\right)\\
&+\int_{\varepsilon<s<s_1<\cdots<s_n<t}\int_{\varepsilon}^{s_1} A_{j+1}(s)\,ds\left(\frac{1}{s_2}I + C_j(s_2)\right)\cdots\left(\frac{1}{s_n}I + C_j(s_n)\right)ds_n \ldots ds_1\\
&= A_{j+1}(\varepsilon)\left(\sum_{k=1}^{\infty}\int_{\varepsilon<s_1<\cdots<s_k<t}\int\left(\frac{1}{s_1}I + C_j(s_1)\right)\cdots\left(\frac{1}{s_k}I + C_j(s_k)\right)ds_k \ldots ds_1\right). \qquad (2.27)
\end{aligned}$$

The almost sure convergence of the series in (2.27) is guaranteed by the fact the process $C_j(s)$, $0 \le s \le T$ is continuous. Since

$$\lim_{\varepsilon \downarrow 0}\frac{1}{\varepsilon}A_{j+1}(\varepsilon) = \sum_{\ell=1}^{r} g^{\ell}(0)\left(g^{\ell}(0)\right)^{\tau}, \qquad (2.28)$$

from(2.27) we infer

$$A_{j+1}(t) = \sum_{\ell=1}^{r} g^{\ell}(0)\left(g^{\ell}(0)\right)^{\tau} \qquad (2.29)$$

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The equality in (2.28) follows from (2.27), which yields the boundedness and continuity of the process $A_{j+1}(t)$, together with the fact that, by induction with respect to j,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} F_j(\varepsilon) = \sum_{\ell=1}^{r} g^{\ell}(0)(g^{\ell}(0))^{\tau}.$$

Finally we let j tend to ∞ to obtain (2.23), with

$$(V_0(t), F_0(t)) = \lim_{j \to \infty} (V_{j+1}(t), F_j(t)).$$

Remarks:

1. All the results which we have established in this work can be extended by replacing the Brownian motion W by another process$Z: [0, a] \times \Omega \to \mathbf{R}^{\mathbf{d}}$, ($d \in \mathbf{N}$) which is a continuous martingale adapted to $\{\mathcal{F}_t\}_{t \in [0,a]}$ and has independent increments and satisfies with some constant K the inequalities
$|Z(t) - Z(s)\mathcal{F}_s| \leq K(t-s)$ and
$\mathbf{E}(|Z(t) - Z(s)|^2 \mathcal{F}_s \leq K(t-s)$for $0 \leq s \leq t \leq a$. Observe that the above properties of Z which we have just mentioned are the only properties of W which we have used (in case of Brownian motion) to prove the results which we have obtained in this work. See , , and .

2. All the lemmas and theorems in this work hold for any delay interval $J' = [-r, 0) \quad (r \geq 0)$ inplace of $J = [-1, 0)$. See , , and .

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